Euclid's Elements - A Survey - Book 1

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1 Definitions - Book 1

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines.
7. A plane surface is (any) one which lies evenly with the straight-lines on itself.
8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
11. An obtuse angle is one greater than a right-angle. 12. And an acute angle (is) one less than a right-angle. 13. A boundary is that which is the extremity of something.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.
18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.
19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.
21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.
22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.
23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direc- tion, meet with one another in neither (of these directions).

2 On Axiomatic Systems

Euclid's the Elements is a treatise on geometry pursuaded using axiomatic theory or a formal system. When one considers an axiomatic system one is asked to accept certain primitive definitions under some set of reasonable arguments in a philosophy. If these certain axioms can be accepted as reasonable then the purpose of axiomatic theory is to derive from these axioms a broader theory or formal system by way of propositions or theorems associated with proof that are demonstrated to be true by way of proof or infallible argumetation. Through centuries of mathematical discourse we have still as a foundation the method of reason motivated by Euclid. We see set theory, probability theory and statistics, and countless other studies in mathematics founded in axiomatic systems. Consider probability theory founded in (3) certain axioms, namely:

Definition 1. A nonempty collection of subsets $\mathcal{A}$ of a set Ω is called a σ-field of subsets of Ω provided that the following two properties hold:

i. If A is in $\mathcal{A}$, then A^c is also in $\mathcal{A}$.
ii. If A_n is in $\mathcal{A}$, for $n = 1, 2, ...$, then $\cup_{n=1}^{\infty} A_n$ and $\cap_{n=1}^{\infty} A_n$ are both in $\mathcal{A}$.

Definition 2. A probability measure P on a σ-field of subsets $\mathcal{A}$ of a set Ω is a real-valued function having domain $\mathcal{A}$ satisfying teh following properties:

i. $P(\Omega) = 1$
ii. $P(A) \geq 0$ for all $A \in \mathcal{A}$
iii. If A_n, for $n = 1, 2...$, are mutually disjoint sets in $\mathcal{A}$, then $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$

Definition 3. A probability space, denoted $(\Omega, \mathcal{A}, P)$ is a set Ω, a σ-field of subsets $\mathcal{A}$, and a probability measure P defined on $\mathcal{A}$.

Set Theory also has its roots in a few fundamental axioms. More so Axiomatic Set Theory can be used to encompass the body of mathematics we know present day. Our objective in selecting axioms is to select a collection that is essential and complete. Where we say essential as necessary and complete as encompassing all postulates applicable to some certain formal system.

3 Philosophical Considerations

We must consier that for some *thing* to be present in space that *thing* must have volume. Then philosophically we can interpret any *thing* that exists in a single dimension or planar space as *some* form of a *thing*. We must note that *things* need not be objects of space in general however certain *things* have empirical validity when considered in a real space. Where here we are not using the term *real space* formally. Then axiomatically Definition 1. (i.e. A point is that of which there is no part.) with respect to a real space is an element of some *thing*. We can deduce that any *thing* is a *thing iff* it has as an element points. Philosophically, we can further extend our notion of form of a *thing* beyond geometry. Modern day we can consider periodic elemental composition, coordinate position with respect to our world or some other setting implying existence, potential energy, color or saturation, etc. This interpretation of some *thing* via its forms has no necessary bounds with respect to modern philosophy. Yet, in our future we may identify a certain set of forms from which all other forms are derivative. This is a deeper question in the philosphy of nature and spirit.
The Elements pursuades certain *forms* that are observable when studying an intuitive or also called Euclidean geometry. Book 1 leads into discourse on observations within this geometry. The discourse spans at most planar space (i.e. two dimensions be them spacial or some other more abstract setting). Definition 1, that of a point, introduces only subtle paradox, namely where it exists. A point inherently can exist in any space be it planar, real, hyper, or at the other end of the spectrum one of infinite dimension. A point has its place in any such space as it simply presents us the notion of some existence within this space. Upon introducing the line in Definition 2 we give rise to more paradoxical intuitions. Most notable a set of paradoxes explicated by Zeno of Elea. In particular, if we were to consider a continuous line our traversal of this line from origin to destination would be impossible if this line had no certain discontinuity due to the supposition that we must at some point pass the mid point of this respective line. At which point we must travese the midpoint of the line explicated by the first midpoint and our destination. Then we can continuously assume that we have passed a similar midpoint as we traverse. In effect our path converges at some focus prior to our destination under the reasonable assumption that we may continuously pass a subsequent midpoint between us and our destination

on our way to this certain destination. Then our only reasonable interpretation of a real space (i.e. one that we experience) is that fundamentaly we can't assume true continuity. The space must be composed of certain atoms that form a discrete setting†. We in effect have sublty introduced the philosophical difference between discrete and continuous mathematics. When we consider Euclid's definitions, we are more concerned with an a priori interpretation. In other words we must consider both the case a line is composed of some set of discrete points which may be so fine they become negligble or more generally a continuous interprestation which is actually stronger than the discrete interpretation. We then see by way of example that certain considerations arise as we look more deeply at the definitions proposed by Euclid. In particular, we introduce greater paradox as we get closer to a real interpretation of this geometry. Yet also we see that with respect to the paradox of Zeno, we are in a way forced to conclude that reality has a property to it that all things are finite elementaly. Continuum with respect to real *things* is unlikely.

† An exercise in thought that may lead to further paradox is to consider the nature of discrete space when we can without doubt assume that our traversal (i.e. our animation) may be truely continuous.

4 Study of Definitions and Open Discussion

We can see by way of example that even meditation on ancient text leads us to better solidify our intuition applied to modern mathematics. Many of the paradoxical considerations apparent within this ancient text have spurred deep work in rigorizing what we mean formally in mathematics, further labeling mathematics as a study *a priori* contradistinct to physics or natural philosophy which is concenred more closely with *a posteriori* expereince. We will pick up in a proceeding lecture.

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